THE EFFECT OF CERTAIN MATROID OPERATIONS ON THE WHITNEY NUMBERS OF THE FIRST KIND AND THE f-VECTOR

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Abstract. We investigated the conjecture that the chromatic polynomials of simple non-representable matroids are unimodal. We grouped matroids together based on their chromatic polynomial, and looked at which matroidal properties are encoded in the chromatic polynomial. We also investigated how certain matroid operations affect the chromatic polynomial, preferably in a controlled way. In patricular, we were interested in which matroid operations preserve log-concavity and unimodality of the chromatic polynomial and of the closely-related f-vector. Circuit-hyperplane relaxation was of particular interest because it is an operation that often messes up the representability of a matroid. We conjecture that log-concavity of the chromatic polynomial is preserved under circuit hyperplane relaxation. We used the Sage Mathematics software to generate matroids, and to compute their chromatic polynomials and f-vectors.

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1. Introduction

It has been conjectured that the chromatic polynomials of matroids are unimodal. June Huh and Eric Katz showed in their paper “Log-concavity of characteristic polynomials and the Bergman fan of matroids” that the chromatic polynomials of representable matroids are log-concave, and as a consequence unimodal. This project aimed to look at the chromatic polynomials of simple non-representable matroids.

This was approached in two ways. First, we grouped matroids together based on their chromatic polynomial, and investigated why certain matroids share the same chromatic polynomial. In particular, we looked at which matroidal properties are encoded in the chromatic polynomial, and which properties encoded in the Tutte polynomial are lost (i.e. not encoded) in the chromatic polynomial. Families defined by the chromatic polynomial contained both representable and non-representable matroids and were of varying structures. Families consisting of only one matroid (“P-unique” matroids”) were of particular interest.

We also looked for operations that change the matroid but not its chromatic polynomial. However we could not find such operations, and concluded that this approach was not very fruitful.

The second approach was investigating how certain matroid operations affect the chromatic polynomial, preferably in a controlled way. In particular, we were interested in which matroid operations preserve log-concavity and unimodality of the chromatic polynomial. In addition, the preservation of log-concavity in the closely related f-vector was also studied. Circuit-hyperplane relaxation was of particular interest because it is an operation that often messes up the representability of a matroid. We conjecture that log-concavity of the chromatic polynomial is preserved under the circuit hyperplane relaxation of a matroid.

We used the Sage Mathematics software to do matroid computations. In particular, we generated all matroids of ≤ 9 elements and rank ≤ 6, then created several dictionaries of these matroids, where matroids were indexed based on either their chromatic polynomial or f-vector. We also used Sage to do operations on matroids, and to compute the chromatic polynomial and f-vector of matroids.

2. Definitions

Definition. A matrix is representable if it can be written as a column matroid defined by a matrix over a field $\mathbb{F}$. The columns of A correspond to matroid elements and dependent sets of the matroid correspond to linearly dependent columns of the matroid.

Definition. The Tutte polynomial of $M = (E, \mathcal{I})$ is defined as

$$T_M(x,y) = \sum_{A \subseteq E} (x-1)^{r(M)-rk(A)}(y-1)^{|A|-rk(A)}$$

Definition. The combinatorial Möbius function of a lattice, which we need to define the Whitney numbers, is a function μ that satisfies the following conditions:

(i) $\mu(x,x) = 1$

(ii) $\mu(x,z) = -\sum_{x \leq y < z} \mu(x,y)$ if $x < z$

(iii) $\mu(x,z) = 0$ if $x \not\leq z$.

The Whitney numbers of the first kind, denoted by w_i, are $\sum_{a:r(a)=i} \mu(0,a)$ where μ is the combinatorial Möbius function for the partially ordered lattice of flats. The Whitney numbers of the second kind, denoted by W_i, are the number of flats of rank i.

Definition. The chromatic polynomial of M is given by

$$\chi_M(t) = (-1)^{r(M)} T_M(1-t, 0) = \sum_{X \subseteq E} (-1)^{|X|} t^{r(M)-r(X)}.$$

An equivalent definition of $\chi_M(t)$ shown by Brylawski is

$$\chi_M(t) = w_0^+ t^n - w_1^+ t^{n-1} + w_2^+ t^{n-2} - w_3^+ t^{n-3} + \cdots \pm w_{n-2}^+ t^2 \mp w_{n-1}^+ t \pm w_n^+$$

where w_j^+ are the unsigned Whitney numbers of the first kind, i.e. $w_j^+ = |w_j| \geq 0$ for every j. We know that $w_0^+ = 1$.

Definition. In this paper T-unique, P-unique, and F-unique matroids are matroids whose Tutte polynomial, chromatic polynomial, and f-vector respectively are not shared by non-isomporphic matroids.

Lemma 2.1. *$t-1$ is a root of the chromatic polynomial of a matroid M.*

Proof.

$$(-1)^{r(M)}T_M(0,0) = \sum_{A\subseteq E}(-1)^{r(M)-rk(A)}(-1)^{|A|-rk(A)} = \sum_{A\subseteq E}(-1)^{|A|} = \sum_{i=0}^{|E|}\binom{|E|}{i} = 0$$

□

Definition. Hence it is useful to define the reduced chromatic polynomial

$$\overline{\chi}_M(t) = \frac{1}{t-1}\chi_M(t)$$

From [3], log concavity of the reduced polynomial implies log concavity of the chromatic polynomial.

Lemma 2.2. *Log concave implies unimodal.*

Proof. Suppose the sequence $(v_0, v_1, ..., v_r)$ of non-negative real numbers is log-concave, i.e. $v_k^2 \geq v_{k-1}v_{k+1}$ for $k \in [1, r-1]$. The sequence is unimodal if and only if there does not exist k such that $v_k < \min(v_{k-1}, v_{k+1})$.

- Case 1: $v_k \geq v_{k+1}$

$$v_k \geq v_{k+1} = \min(v_{k-1}, v_{k+1}).$$

- Case 2: $v_k < v_{k+1}$

$$\frac{v_k}{v_{k+1}} < 1$$
$$v_{k-1} \leq \frac{v_k}{v_{k+1}}v_k < v_k$$
$$v_{k-1} < v_k < v_{k+1}$$
$$v_k \geq v_{k-1} = \min(v_{k-1}, v_{k+1}).$$

Since this is true for all k, the sequence is unimodal. □

3. Tutte and Chromatic polynomial

From [6] the following invariants of a matroid M on a set E can be deduced from its Tutte polynomial $T_M(x, y)$:

(i) the rank $r(M)$;

(ii) the size $|E|$;

(iii) the number of rank-i sets of cardinality j for each i, j with $0 \leq i \leq r(M)$ and $0 \leq j \leq |E|$

(iv) for each i with $0 \leq i \leq r(M)$, the number of independent sets of M of cardinality i;

(v) for each i with $0 \leq i \leq r(M)$, the largest cardinality among all flats of M of rank i, and the number of rank-i flats of this cardinality;

(vi) the girth $g(M)$, which is the cardinality of the smallest circuit, and the number of circuits of M that have cardinality $g(M)$.

Only the first two of the items in this list are invariants of the chromatic polynomial. As a counterexample, consider the matroids M_1 and M_2 of rank 4, both on the groundset $E = \{0, 1, 2, 3, 4, 5, 6\}$ with the following circuit closures:

- The circuit closures of M_1:
  - Rank 2: $[0,1,2]$, $[1,3,5]$, $[0,4,5]$, $[2,3,4]$
  - Rank 3: $[0,1,2,3,4,5]$
- The circuit closures of M_2:
  - Rank 2: $[0,1,2,3]$, $[4,5,6]$

Then by simple computations on Sage, the following can be verified:

- M_1 and M_2 have the same chromatic polynomial,
$$\chi_{M_1}(t) = \chi_{M_2}(t) = t^4 - 7t^3 + 17t^2 - 17t + 6.$$
- M_1 has 31 independent sets of cardinality 3, while M_2 has only 30 independent sets of cardinality 3. Also, M_1 has 16 independent sets of cardinality 4, while M_2 has 18 independent sets of cardinality 4.
- M_1 has 30 independent sets of cardinality 3, while M_2 has 31 independent sets of cardinality 3. Also, M_1 has 16 independent sets of cardinality 4, while M_2 has 18 independent sets of cardinality 4.
- M_1 has 13 rank-2 flats of cardinality 3, which is the largest cardinality among all flats of M_1 of rank 2, whereas M_2 has 14 rank-2 flats of cardinality 4, which is the largest cardinality among all flats of M_2 of rank 2.
- $g(M_1) = 4 \neq 3 = g(M_2)$. Also, M_1 has 3 circuits of cardinality $g(M_1)$, while M_2 has 5 circuits of cardinality $g(M_2)$.

This shows that the chromatic polynomial does not encode (iii)-(vi).

In addition, whether M is simple can be deduced from $T_M(x,y)$. Furthermore, if M is simple, the number of 4-circuits of M can also be deduced from $T_M(x,y)$.

(vii) the Tutte polynomial of the dual matroid M^*; in particular, $T_{M^*}(x,y) = T_M(y,x)$.

(viii) M is the direct sum of M_1 and M_2 if and only if $T_M(x,y) = T_{M_1}(x;y)T_{M_2}(x;y)$.

(ix) M is disconnected if and only if $T_M(x,y)$ has a nontrivial factor in $\mathbb{Z}[x,y]$.

(x) If M has c connected components $M1, \cdots, Mc$ then the factorization of $T_M(x,y)$ in $\mathbb{Z}[x,y]$ is $T_M(x;y) = T_{M1}(x;y)\cdots T_{Mc}(x;y)$.

(xi) A matroid is T-unique if and only if each of its connected components is T-unique.

4. P-unique matroids

We present certain classes of matroids that have unique chromatic polynomials.

4.1. **Uniform matroids.**

Definition. The uniform matroid $U_{r,n}$ is the set of n elements whose subsets are independent if and only if it contains at most r elements. A subset is a basis if it has exactly r elements.

Lemma 4.1. *The chromatic polynomial of the uniform matroid $U_{r,n}$ is*
$$t^r - \binom{n}{1}t^{r-1} + \binom{n}{2}t^{r-2} + \cdots + (-1)^{r-1}\binom{n}{r-1}t + \sum_{k=r}^{n}(-1)^k\binom{n}{k}$$

Lemma 4.2. *$\sum_{j=i+1}^{k}\binom{k}{j}(-1)^j$ is negative for i even and positive for i odd where $k > i$.*

Proof. Case 1: $i \leq \frac{k}{2}$
The binomial coefficients $\binom{k}{0},\binom{k}{1},\binom{k}{2}\cdots\binom{k}{\lfloor\frac{k}{2}\rfloor}$ are increasing. $\sum_{j=0}^{i}\binom{k}{j}(-1)^j$ is positive for i even and negative for i odd. Since $\sum_{j=0}^{k}\binom{k}{j}(-1)^j = 0$ this implies that $\sum_{j=i+1}^{k}\binom{k}{j}(-1)^j$ is negative for i even and positive for i odd.
Case 2: $i > \frac{k}{2}$
Then $k-i < \frac{k}{2}$ and $\sum_{j=0}^{k-i-1}\binom{k}{j}(-1)^j$ is positive for $k-i-1$ even and negative for $k-i-1$ odd. Since the binomial coefficents are symmetric this is equal to $(-1)^k\sum_{j=i+1}^{k}\binom{k}{j}(-1)^j$. Hence $\sum_{j=i+1}^{k}\binom{k}{j}(-1)^j$ is positive for $i+1$ even and negative for $i+1$ odd. □

Theorem 4.3. *The chromatic polynomial of the uniform matroid $U_{r,n}$ is unique.*

Proof. Suppose not. Let $U_{r,n}$ and non-isomorphic matroid M have chromatic polynomial $p(t)$.
Let the smallest rank among the circuits of M be i. Since M is not isomorphic to $U_{r,n}$, we have $i < r$. Let A_j be the circuits of rank i and cardinality j. Then the coefficient of t^{r-i} of p is

$$\binom{n}{i}(-1)^i = \binom{n}{i}(-1)^i + \sum_{j=i+1}^{n}\sum_{k=i+1}^{j}\binom{j}{k}(-1)^k|A_j|$$

However, $\sum_{k=i+1}^{j}\binom{j}{k}(-1)^k$ has the same sign independent of j so this is not possible. $U_{r,n}$ is p-unique. □

4.2. **Almost-uniform matroids.**

Definition. An almost-uniform matroid is a matroid with elements $e, f, g \in E(M)$ such that $\{e, f, g\}$ is a circuit and $M\backslash e$ is uniform. Almost-uniform matroids are P-unique as well.

5. Chromatic Polynomial

We write the chromatic polynomial of M as

$$\chi_M(t) = w_0^+t^n - w_1^+t^{n-1} + w_2^+t^{n-2} - w_3^+t^{n-3} + \cdots \pm w_{n-2}^+t^2 \mp w_{n-1}^+t \pm w_n^+$$

where w_j^+ are the unsigned Whitney numbers of the first kind. We know that $w_0^+ = 1$.

5.1. **Matroid operations.** Here we discuss some matroid operations – relaxation, free extension, truncation, parallel connection, generalized parallel connection, direct sum, and two-sum – and their effects on the chromatic polynomial.

Relaxation is a particularly interesting operation for our research problem because it often messes up the representability of a matroid. Other operations can be used to create bigger matroids: free extension and coextension are ways to add an element to a matroid; parallel-connection, generalized parallel-connection, direct sum, two-sum, and three-sum are "gluing" operations that combine matroids to create a bigger matroid. Truncation is an operation that can only increase the connectivity of a matroid, and can be used in conjunction with the above operations to create a bigger matroid with higher connectivity.

5.1.1. *Relaxation.* Let M be a matroid with a circuit hyperplane $X \subseteq E$. Then the matroid M' obtained from M by relaxation is the matroid with basis $B(M') = B(M) \cup \{X\}$.

Proposition 5.1. (Relaxation.) *Let M' be the relaxation of M. Then*
$$\chi_{M'}(t) = t^{n-1} - w_1^+ t^{n-2} + w_2^+ t^{n-2} - w_3^+ t^{n-4} + \cdots \pm w_{n-2}^+ t^2 \mp \big(w_{n-1}^+ + 1\big)t \pm \big(w_n^+ + 1\big).$$

Proof. This follows from the formula given by [1] for the Tutte polynomial of M':
$$T_{M'}(x,y) = T_M(x,y) - xy + x + y.$$
□

Observation. If the chromatic polynomial of M is log-concave, then the chromatic polynomial of M' is unimodal.

Proof. This observation comes from the desire to show that circuit hyperplane relaxation preserves unimodality. It is clear that if χ_M is unimodal and $w_{n-1}^+ + 1 \leq w_{n-2}^+$ that $\chi_{M'}$ is unimodal as well. It suffices to look at the case where $w_{n-1}^+ = w_{n-2}^+$.

It is not generally true that if two consecutive terms in a unimodal sequence are equal then the terms equal to the maximum value of the sequence. Imposing the additional restriction that χ_M is log-concave however forces this to be the case. Unimodality of the chromatic polynomial after relaxation follows. □

5.1.2. *Free extension.* The free extension of a matroid M by an element $e \notin E(M)$ consists of adding e as freely as possible without increasing the rank. We do this by adding e to every circuit closure of full rank.

Proposition 5.2. (Free extension.) *Let M' be the free extension of M. Then*
$$\chi_{M'}(t) = t^n - (w_0^+ + w_1^+)t^{n-1} + (w_1^+ + w_2^+)t^{n-2} - \cdots \pm (w_{n-2}^+ + w_{n-1}^+)t \mp w_{n-1}^+.$$

Proof. This follows from the formula given by [1] for the Tutte polynomial of M':
$$T_{M'}(x,y) = \frac{x}{x-1} T_M(x,y) + \left(y - \frac{x}{x-1}\right) T_M(1,y).$$
□

Proposition 5.3. *Free extension preserves unimodality and log-concavity of the chromatic polynomial.*

Proof. Suppose $\chi_M(t)$ is unimodal. Let a_k be the unimodal "peak", i.e.
$$w_0^+ \leq w_1^+ \leq \cdots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \cdots \geq w_{n-1}^+ \geq w_n^+.$$
Then $a_{j-2} \leq a_j$ for $j \in [2,k]$, so that
$$w_0^+ \leq (w_0^+ + w_1^+) \leq \cdots \leq (a_{j-1} + a_{j-2}) \leq (a_j + a_{j-1}) \leq \cdots \leq (a_k + a_{k-1}).$$

For $j \in [k, n-2]$, since $a_j \geq a_{j+2}$, we either have that
$$(a_k + a_{k-1}) \geq (a_{k+1} + a_k) \geq \cdots \geq (a_{j+1} + a_j) \geq (a_{j+2} + a_{j+1}) \geq \cdots \geq (w_n^+ + w_{n-1}^+) \geq w_n^+$$
or
$$(a_k + a_{k-1}) \leq (a_{k+1} + a_k) \geq \cdots \geq (a_{j+1} + a_j) \geq (a_{j+2} + a_{j+1}) \geq \cdots \geq (w_n^+ + w_{n-1}^+) \geq w_n^+.$$
In either case, we see that the absolute value of the coefficients of $\chi_{M'}(t)$ form a unimodal sequence.

Suppose $\chi_M(t)$ is log-concave. Then since $a_j^2 \geq a_{j-1}a_{j+1}$ and $a_{j-1}^2 \geq a_{j-2}a_j$, multiplying the two inequalities gives us

$$a_{j-1}^2\ a_j^2 \geq a_{j-2}\ a_{j-1}\ a_j\ a_{j+1}.$$

We divide both sides by $a_{j-1}\ a_j$:

$$a_{j-1}\ a_j \geq a_{j-2}\ a_{j+1}.$$

Again, using the fact that $a_j^2 \geq a_{j-1}a_{j+1}$ and $a_{j-1}^2 \geq a_{j-2}a_j$,

$$a_j^2 + a_{j-1}\ a_j + a_{j-1}^2 \geq a_{j-1}\ a_{j+1} + a_{j-2}\ a_{j+1} + a_{j-2}\ a_j$$

Adding $a_{j-1}a_j$ to both sides and factoring, we get the desired inequality

$$(a_j + a_{j-1})^2 \geq (a_{j-1} + a_{j-2})(a_{j+1} + a_j).$$

Since this is true for any j, we conclude that $\chi_{M'}(t)$ is log-concave. □

5.1.3. *Truncation.* The truncation of M is the matroid obtained by adding an element freely and then contracting that element.

Proposition 5.4. (Truncation.) *Let M' be the truncation of M. Then*

$$\chi_{M'}(t) = t^{n-1} - w_1^+ t^{n-2} + w_2^+ t^{n-2} - w_3^+ t^{n-4} + \cdots \pm w_{n-2}^+ t \mp (w_{n-1}^+ - w_n^+).$$

Proof. This is easy to verify from the explicit formula. □

Proposition 5.5. *Truncation preserves unimodality and log-concavity of the chromatic polynomial.*

To prove the above proposition, we use the following lemma:

Lemma 5.6. *For matroids M with $r(M) \geq 2$, the sequence of the absolute value of the coefficients of $\chi_M(t)$ is not monotonically increasing.*

Proof. Since 1 is a root of $\chi_M(t)$, we must have that

$$w_0^+ - w_1^+ + w_2^+ - w_3^+ + \cdots \mp w_{n-1}^+ \pm w_n^+ = 0.$$

Suppose by contradiction that $\chi_M(t)$ is monotonically increasing, i.e. $a_{j-1} \leq a_j\ \forall j$.. Then if n is even,

$$\begin{aligned} 0 &= w_0^+ - w_1^+ + w_2^+ - w_3^+ + \cdots \mp w_{n-1}^+ \pm w_n^+ \\ &= w_0^+ + (-w_1^+ + w_2^+) + (-w_3^+ + w_4^+) + \cdots + (-w_{n-1}^+ + w_n^+) \\ &> 0. \end{aligned}$$

and if n is odd,

$$\begin{aligned} 0 &= 1 - w_1^+ + w_2^+ - w_3^+ + \cdots \pm w_{n-1}^+ \mp w_n^+ \\ &= (1 - w_1^+) + (w_2^+ - w_3^+) + \cdots + (w_{n-1}^+ - w_n^+) \\ &< 0. \end{aligned}$$

In either case we get a contradiction, so the coefficients of $\chi_M(t)$ cannot be monotonically increasing. □

Proof of Proposition 5.5. Let M' be the matroid obtained by truncating M. Then its chromatic polynomial is of the form

$$\chi_{M'}(t) = t^{n-1} - w_1^+ t^{n-2} + w_2^+ t^{n-2} - w_3^+ t^{n-4} + \cdots \pm w_{n-2}^+ t \mp (w_{n-1}^+ - w_n^+).$$

By Lemma 5.6 we have that $w_{n-1}^+ \geq w_n^+$, so

$$w_{n-1}^+ \geq (w_{n-1}^+ - w_n^+) \geq 0. \tag{5.1}$$

So if $\chi_M(t)$ is log-concave, then

$$\begin{aligned}(w_{n-2}^+)^2 &\geq w_{n-3}^+ w_{n-1}^+ \\ &\geq w_{n-3}^+ (w_{n-1}^+ - w_n^+)\end{aligned}$$

so that $\chi_{M'}(t)$ must also be log-concave.

Suppose $\chi_M(t)$ is unimodal. Then

- Case 1: $w_{n-2}^+ \leq w_{n-1}^+$ (i.e. w_{n-1}^+ is the unimodal "peak" of $\chi_M(t)$.) Then $\chi_{M'}(t)$ is unimodal.
- Case 2: $w_{n-2}^+ \geq w_{n-1}^+$ (i.e. w_{n-1}^+ is not the unimodal "peak" of $\chi_M(t)$.) Then $w_{n-2}^+ \geq (w_{n-1}^+ - w_n^+)$ by Lemma 5.6, so that $\chi_{M'}(t)$ is unimodal.

□

5.1.4. *Generalized Parallel Connection.* [] Let M_1 and M_2 be simple matroids on the ground sets E_1 and E_2 respectively, such that the following conditions hold:

(i) $M_1|T = M_2|T$, where $T = E_1 \cap E_2$

(ii) T is a modular flat of M_1.

Let $N = M_1|T(= M_2|T)$. Then the generalized parallel connection of M_1 and M_2 at T is the matroid $P_N(M_1, M_2)$ whose flats are the subsets of $E_1 \cup E_2$ of the form $A_1 \cup A_2$, where A_1 and A_2 are flats of M_1 and M_2, respectively.

Proposition 5.7. (Generalized Parallel Conection.)

$$\chi_{P_N(M_1,M_2)}(t) = \frac{\chi_{M_1}(t)\chi_{M_2}(t)}{\chi_N(t)}.$$

5.1.5. *Direct Sum.* For n matroids, call them $M_i = (E_i, I_i)$ for each $i \in [1, n]$, the direct sum $M_1 \oplus M_2 \oplus \cdots \oplus M_n$ is the matroid $(E, \mathcal{I})$ with $E = E_1 \cup E_2 \cup \cdots \cup E_n$ and $\mathcal{I} = I_1 \cup I_2 \cup \cdots \cup I_n$.

Proposition 5.8. (Direct Sum.) *Let $M' = M_1 \oplus M_2 \oplus \cdots \oplus M_n$. Then*

$$\chi_{M'}(t) = \prod_{i=1}^{n} P(M_i; t).$$

Proof. This follows from the well-known direct sum formula for the Tutte polynomial [1],

$$T_{M_1 \oplus M_2 \oplus \cdots \oplus M_n}(x, y) = \prod_{i=1}^{n} T_{M_i}(x, y).$$

□

Proposition 5.9. *Direct sum preserves log-concavity of the chromatic polynomial.*

Proof. If two polynomials p_1 and p_2 are log-concave, then so does the product p_1p_2. (See [2].) □

5.1.6. *Two-sum.* The two-sum of matroids $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$ at a point $x \in E_1 \cap E_2$ is the matroid obtained by first taking the parallel connection of M_1 and M_2 at x, then deleting x.

Proposition 5.10. (Two-sum.) *Let M_1 and M_2 be matroids with $E(M_1) \cap E(M_2) = \{p\}$, where p is not a loop or a coloop in M_1 and M_2. If $M' = M_1 \oplus_2 M_2$, then*

$$\chi_{M'}(t) = \frac{1}{t-1}(p_1(t \cdot p_3 + p_4) + p_2(p_3 + p_4))$$

where

$$\begin{aligned} p_1 &= (-1)^{r(M_1/p)}\chi_{M_1/p} \\ p_2 &= (-1)^{r(M_1\backslash p)}\chi_{M_1\backslash p} \\ p_3 &= (-1)^{r(M_2/p)}\chi_{M_2/p} \\ p_4 &= (-1)^{r(M_2\backslash p)}\chi_{M_2\backslash p}. \end{aligned}$$

Proof. This follows from the formula [1]

$$T_{M_1 \oplus_2 M_2} = \frac{1}{xy - x - y} \begin{bmatrix} T_{M_1/p} & T_{M_1\backslash p} \end{bmatrix} \begin{bmatrix} x-1 & -1 \\ -1 & y-1 \end{bmatrix} \begin{bmatrix} T_{M_2/p} \\ T_{M_2\backslash p} \end{bmatrix}.$$

□

6. F-vector

Let M be a matroid of rank r. Let L_F be the lattice of flats of M, and let L_I be the lattice of indpendent sets of M. Let $f = (f_0, ..., f_r)$ be the f-vector of L_I, i.e. f_i is the number of sets of cardinality i in L_I. The f-polynomial is the polynomial

$$f_M(q) = \sum_{i=0}^{r} f_i q^{r-i},$$

and the h-vector $(h_0, ..., h_r)$ consists of the coefficients of the h-polynomial defined by

$$h_M(q) = \sum_{i=0}^{r} h_i q^{r-i} = f_M(q-1).$$

It is known that log-concavity of h-vectors implies strict log-concavity of f-vectors (See [3]).

6.1. **The existence of nonunimodal f-vectors.**

Remark. It has recently been proven that not all pure f-vectors are unimodal (See [5], Theorem 2.2). Pastine and Zanello show that, for any $N \geq 2$, there exists a nonunimodal pure f-vector having N peaks. However, their construction of the nonunimodal f-vector involves taking the disjoint union of many lattices. Since the disjoint union of lattices of independent sets, or of flats, of a matroid does not define a valid matroid, this nonunimodal f-vector is irrelevant for matroids.

6.2. **Matroid operations.**

6.2.1. *Free coextension.* There is an interesting link between the f-vector and the Whitney numbers of the first kind, which was proved by Brylawski in 1977 [4].

Theorem 6.1. *Let M be a matroid of rank r, and let $M \times e$ be its free coextension. Then*

$$(-1)^{r+1}\chi_{M\times e}(-q) = (1+q)f_M(q)$$

Corollary 6.2. *If matroids M_1 and M_2 have the same f-vector, then their free coextensions have the same $\chi(t)$.*

6.2.2. *Free extension.*

Observation. Let f' be the f-vector of the free extension of M. Then

$$f' = (f_0,\ f_0+f_1,\ f_1+f_2,\ ...,\ f_{r-1}+f_r).$$

6.2.3. *Relaxation.*

Observation. Let f' be the f-vector of the relaxation of M. Then

$$f' = (f_0,\ f_1,\ f_2,\ ...,\ f_{r-1},\ f_r+1).$$

6.2.4. *Truncation.*

Observation. Let f' be the f-vector of the truncation of M. Then

$$f' = (f_0, f_1, ..., f_{r-2}, f_{r-1}).$$

6.2.5. *Direct Sum.*

Observation. For any matroids M_1 and M_2,

$$f_{M_1\oplus M_2}(q) = f_{M_1}(q)f_{M_2}(q).$$

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