COXETER MATROIDS

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1. Introduction

Coxeter matroids, which are based on a Coxeter group W and a standard parabolic subgroup P of W, are a generalization of ordinary matroids[1]. Associated with each Coxeter matroid is the Coxeter matroid polytope, which turns out to be an equivalent way to define Coxeter matroids.

In this paper, we will focus primarily on the Coxeter matroids for the Coxeter groups $W = \mathcal{A}_{n-1} = S_n$ (the symmetric group) and $W = BC_n$ (the hyperoctahedral group).

1.1. **The symmetric group.** S_n is the group whose elements are all the permutations of $E = [n]$, and is generated by the set of adjacent transpositions, $\{s_1, s_2, ..., s_{n-1}\}$ where $s_i := (i, i+1)$.

1.2. **The hyperoctaheral group.** BC_n is the group of symmetries of the n-cube. We can represent BC_n as a permutation on the set $E = \{1, 2, ..., n, 1^*, 2^*, ..., n^*\}$, corresponding to the facets of the n-cube (with i the facet opposite i^*). Then BC_n is generated by the involutions $s_1 := (1, 2)(1^*, 2^*)$, $s_2 := (2, 3)(2^*, 3^*)$, ..., $s_{n-1} := (n-1, n)((n-1)^*, n^*)$, $s_n := (n, n^*)$.

2. Coxeter Matroid

2.1. **Induced Bruhat ordering.** Let $\leq$ denote the Bruhat order[2] on W. The *w-Bruhat order* (or *shifted Bruhat order with center w*) $\leq^w$ is defined as follows: $u \leq^w v$ means that $w^{-1}v \leq w^{-1}u$.

Bruhat order on W induces a well-defined partial order (also called Bruhat order) on the factor set $W^P = W/P$, where $tP \leq rP$ if there exist $s \in tP$, $q \in rP$ such that $s \leq q$ in ordinary Bruhat order. For two cosets $A, B \in W^P$ and element $w \in W$, $A \leq^w B$ means that $w^{-1}A \leq w^{-1}B$.

2.2. **Parabolic subgroup.** Let Π be a simple system in the root system Φ with corresponding system of reflections $r_1, ..., r_m$. For a subset $I \subseteq [m]$, the subgroup

$$W_I := \langle r_i : i \in I \rangle$$

is called a *standard parabolic subgroup* of W. A subgroup conjugate in W to a standard parabolic subgroup is called *parabolic*.

2.3. **Coxeter matroid.** Let W be a Coxeter group, and P a standard parabolic subgroup of W. We define $\mathcal{M} \subseteq W^P$ to be a *Coxeter matroid* if $\mathcal{M}$ satisfies the *Maximality Property*:

> For any $w \in W$, there is a unique $A \in \mathcal{M}$ such that, for all $B \in \mathcal{M}$, $B \leq^w A$.

Although the Maximality Property is a nice, clean condition from a theoretical perspective, it is rather difficult to check directly, even on small examples. Fortunately, the Gelfand-Serganova theorem (presented later) provides an equivalent geometric condition which is easier to check.

[1]For the definition of an ordinary matroid, please see Appendix A.

[2]The Bruhat order $\leq$ is defined as follows. Let $u, v \in W$. Then $u \leq v$ if and only if there exists a reduced expression $v = s_{i_1} s_{i_2} \cdots s_{i_p}$ for v such that $u = s_{i_{j_1}} s_{i_{j_2}} \cdots s_{i_{j_q}}$ for some $j_1, j_2, ..., j_q$ with $1 \leq j_1 < j_2 < \cdots < j_q \leq p$.

2.4. **Ordinary matroids.** In the special case when $W = S_n$ and P is a maximal[3] parabolic subgroup, Coxeter matroids are equivalent to ordinary matroids. Here we give an example for $W = S_4$.

2.4.1. *Example.* Let $P = W_{\{1,3\}} = \langle s_1, s_3 \rangle$ be a maximal standard parabolic subgroup of S_4. Thus

$$P = <(12), (34)> = \{1234, 2134, 1243, 2143\}$$

where the last representation comes from thinking of the permutations in S_4 as permuting the word 1234, hence this word itself represents the identity element of the group. Similarly, the elements of a left coset may be represented by words, such as

$$(23)P = \{1324, 3124, 1342, 3142\}.$$

Every left coset of P corresponds to a 2-element subset of E, namely, the two elements which appear, in some order, as the first two symbols in each word of the coset. For example, P corresponds to the subset $\{1, 2\}$, and $(23)P$ corresponds to the subset $\{1, 3\}$. Thus a Coxeter matroid in S_4^P may be represented by a collection of 2-element subsets of E which must, obviously, correspond to cosets satisfying the Maximality Property.

For example, let $\mathcal{M} = \{12, 13, 14, 23, 24\}$, where ab is an abbreviation of the two-element subset $\{a, b\}$ of E. Each pair listed corresponds to a left coset of P, namely

$$\begin{array}{lcl} \{1,2\} & \leftrightarrow & P = \{1234, 2134, 1243, 2143\} \\ \{1,3\} & \leftrightarrow & (23)P = \{1324, 3124, 1342, 3142\} \\ \{1,4\} & \leftrightarrow & (24)P = \{1432, 4132, 1423, 4123\} \\ \{2,3\} & \leftrightarrow & (13)P = \{3214, 2314, 3241, 2341\} \\ \{2,4\} & \leftrightarrow & (14)P = \{4231, 2431, 4213, 2413\}. \end{array}$$

Using the Gelfand-Serganova theorem, we will later prove in Section 4.1 that $\mathcal{M}$ is a Coxeter matroid. But also, $\mathcal{M}$ can be realized as an ordinary matroid[4] of rank 2 on $E = [4]$ whose bases $\mathcal{B}$ consist of all 2-element subsets except $\{3, 4\}$.

3. A more geometric perspective

Associated with each finite Coxeter group W is the *Coxeter complex*, a cell complex whose chambers are in bijection with the elements of W. The geometric concepts associated with the Coxeter complex form the language of the theory of Coxeter matroids. For example, as mentioned above, the Gelfand-Serganova theorem provides an equivalent geometric condition for the Maximality Property, by interpreting Coxeter matroids as *Coxeter matroid polytopes.*

3.1. **Terminology.** A finite set Σ of hyperplanes in the affine space $\mathbb{AR}^n$ is called a *hyperplane arrangement.* We call hyperplanes in Σ *walls* of Σ.

The hyperplanes in Σ cut the space $\mathbb{AR}^n$ and each other in pieces called faces. More formally, a hyperplane cuts the space $\mathbb{AR}^n$ into two halfspaces V^+ and V^-. For two points $a, b \in \mathbb{AR}^n$, we write $a \sim b$ if, for each hyperplane $H \in \Sigma$, a and b belong to one and the same of two half spaces V^+, V^- determined by H. Obviously $\sim$ is an equivalence relation. Its equivalence classes are called *faces* of Σ.

A face of Σ that is not contained in any hyperplane of Σ is called a *chamber.* A *facet* of a chamber C is a face of dimension $n - 1$ on the boundary of C. It follows from the definition that a facet P belongs to a unique hyperplane $H \in \Sigma$, called a *wall* of the chamber C. (For an example, see Fig. 1.)

[3]A parabolic subgroup W_I is *maximal* if $I \subset [m]$ is obtained by discarding one index in $[m]$.

[4]To see this, we can check the basis axioms for ordinary matroids (see Appendix A).

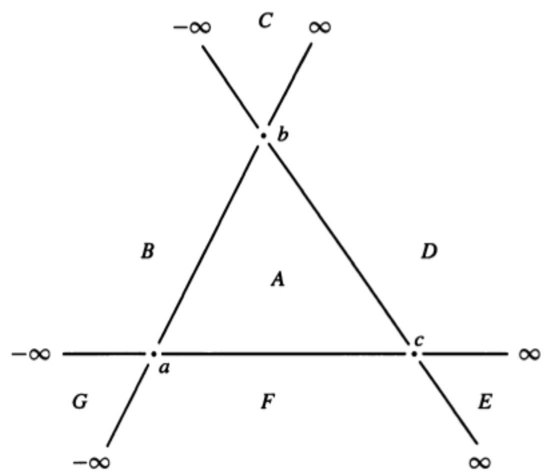


FIGURE 1. The three lines divide $\mathbb{AR}^2$ into seven open faces $A, ..., G$ (chambers), nine 1-dimensional faces $(-\infty, a), (a, b), ..., (c, \infty)$, and three 0-dimensional faces a, b, c. A facet of the chamber A is the 1-dimensional interval (a, b), which belongs to the unique line passing through a and b.

3.2. **The Coxeter complex.** Let Φ be a root system in the euclidean space V, and fix a simple system $\Pi = \{\alpha_1, ..., \alpha_m\}$ with corresponding system of simple reflections $r_1, ..., r_m$.

A wall H_α corresponding to a root α cuts the space V into two open half-spaces $V_\alpha^+ = \{\lambda \in V : (\lambda, \alpha) > 0\}$ and $V_\alpha^- = \{\lambda \in V : (\lambda, \alpha) < 0\}$. The fixed hyperplanes of all the reflections give a hyperplane arrangement, with each root $\alpha \in \Phi$ being orthogonal to one of the hyperplanes in the arrangement. Since all the hyperplanes intersect at the origin, the chambers are open polyhedral cones[5] (see Fig. 2). Let $\mathcal{W}$ be the set of all chambers associated with the root system Φ.

The chamber $E = \bigcap\limits_{\alpha \in \Pi} V_\alpha^+$ is the fundamental chamber of $\mathcal{W}$ (with respect to Π). Thus

$$E = \{\lambda \in V : (\lambda, \alpha) > 0 \quad \forall \alpha \in \Pi\}.$$

The group W is simply transitive on $\mathcal{W}$, i.e. for any two chambers C and D in $\mathcal{W}$ there exists a unique element $w \in W$ such that $D = wC$ (See [1], Thm 5.7.3). Moreover, the map

$$w \mapsto wE$$

is a one-to-one correspondence between the elements in W and chambers in $\mathcal{W}$ (See [1], Thm 5.8.2).

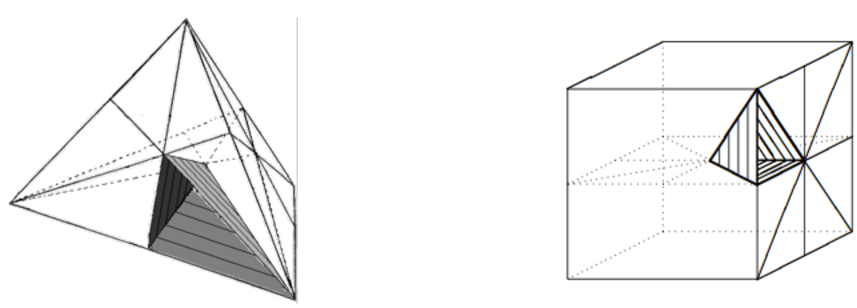


FIGURE 2. A chamber in the Coxeter complex of S_4 (left) and of BC_3 (right).

[5]A *polyhedral cone* is a cone which is an intersection of finitely many closed half-spaces, with the origin 0 belonging to the bounding hyperplane of each of these half-spaces.

For each subset $I \subseteq [m]$, define

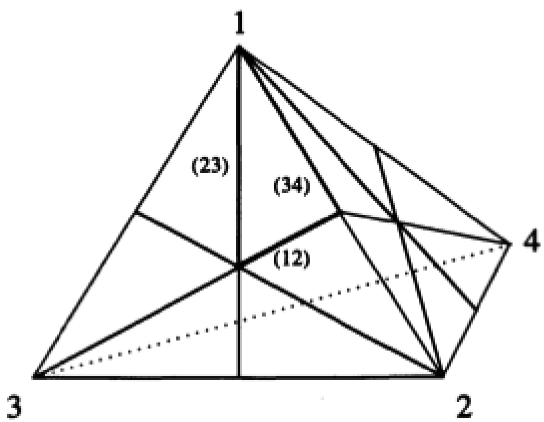
$$C_I := \left\{ \lambda \in V : \begin{array}{ll} (\lambda, \alpha_i) = 0 & \forall i \in I \\ (\lambda, \alpha_i) > 0 & \forall i \notin I \end{array} \right\}.$$

Thus C_I is an intersection of certain hyperplanes H_α and certain open half-spaces V_α^+, and is a face of the complex. It is clear that the sets C_I partition $\overline{E}$, with $C_\emptyset = E$ and $C_{[m]} = \{0\}$.

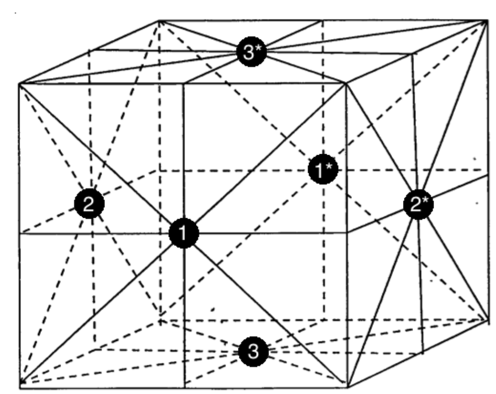
Since W is simply transitive on the set $\mathcal{W}$ of all chambers, it follows that V is partitioned by the collection $\mathcal{C}$ of all sets wC_I ($w \in W, I \subset [m]$). More precisely, for each fixed $I \subset [m]$, the sets wC_I and $w'C_I$ are disjoint unless w and w' lie in the same left coset in W/W_I, in which case they coincide. For distinct I and J, all sets wC_I and $w'C_J$ are disjoint. We call $\mathcal{C}$ the *Coxeter complex* of W. Note that $\mathcal{W} \subset \mathcal{C}$. Any set wC_I is called a face of *type* I.

3.2.1. *Examples.* The Coxeter complex of S_n is the barycentric subdivision of the regular $n-1$-dimensional simplex, and its roots are parallel to the edges of the simplex (see Fig. 3a).

A Coxeter matroid for $W = BC_n$ is called a *symplectic* matroid. Its Coxeter complex is the barycentric subdivision of the n-cube, and its roots are parallel to the edges of the n-cube and to the diagonals of the 2-dimensional faces of the n-cube (see Fig. 3b).



(A) The Coxeter complex of S_4, which is generated by the hyperplanes of symmetry of the tetrahedron. (Recall that S_4 is isomorphic to the octahedral group which acts on its 4 vertices.)



(B) The Coxeter complex of BC_3, which is generated by the hyperplanes of symmetry of the cube.

FIGURE 3. The Coxeter complexes of S_n and BC_n. Here, the hyperplanes of all the reflections are shown by their lines of intersection with the faces of the simplex and the cube.

3.3. **Parabolic subgroups and faces of type I.**

The characterization of parabolic subgroups as isotropy groups yields a geometric interpretation of Coxeter matroids.

Proposition 3.3.1. *For $I \subset [m]$, the isotropy group[6] of C_I is precisely the parabolic subgroup W_I.*

[6]If $X \subset V$ is a set of vectors, then its *isotropy group* $C_W(X)$ is the group

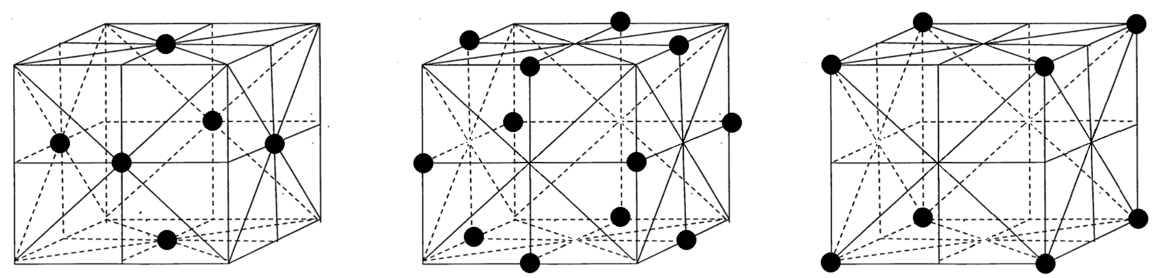
$$C_W(X) = \{w \in W : w\lambda = \lambda \quad \forall \lambda \in X\}$$

Then it immediately follows that the cosets of the parabolic subgroup W_I are in bijection with the faces of the complex of type I. So if $\mathcal{M}$ is any subset of the cosets of W_I in W, then $\mathcal{M}$ corresponds to a collection of faces of type I.

3.3.1. *Faces of type I.* For a subset $I \subset [m]$ obtained by removing one index from $[m]$, let $\xi(I)$ be the unique index in $[m]$ omitted from I.

The vertices of the complex $\mathcal{W}$ correspond to the left cosets wW_I, where W_I is a maximal parabolic subgroup. In the case of S_n, each vertex of type I is the barycenter of a face of cardinality $\xi(I)$ of the simplex. In the case of BC_n, each vertex of type I is the barycenter of a face of the cube of dimension $n - \xi(I)$ (see Fig. 4).

The type of a larger dimensional face is just the intersection of the types of its vertices. For example, in the Coxeter complex of BC_3, any edge connecting the barycenter of an edge of the cube with a vertex of the cube would be a face of $\mathcal{W}$ of type $\{1\}$, whereas an edge connecting an edge-barycenter to a face-barycenter would be a face of $\mathcal{W}$ of type $\{3\}$.



(A) Faces of type $\{2, 3\}$ (B) Faces of type $\{1, 3\}$ (C) Faces of type $\{1, 2\}$

FIGURE 4. The faces corresponding to a maximal parabolic subgroup W_I of $W = BC_3$

3.4. **The polytope associated with $\mathcal{M}$.** For a subset $\mathcal{M} \subseteq W^P$, let $\delta(\mathcal{M})$ denote the set of barycenters of the faces of $\mathcal{W}$ which correspond to $\mathcal{M}$. Note that if P is a maximal parabolic subgroup, then $\delta(\mathcal{M})$ is just a subset of the vertices of $\mathcal{W}$, but if P is non-maximal, then we are actually taking barycenters of faces of $\mathcal{W}$, which is itself the barycentric subdivision of the simplex or cube. Now we define the *polytope*[7] of $\mathcal{M}$, or $\Delta_{\mathcal{M}}$, to be the convex hull of points in $\delta(\mathcal{M})$.

$\mathcal{M}$ is a Coxeter matroid if and only if $\Delta_{\mathcal{M}}$ is a *Coxeter matroid polytope.* (For a formal definition of a Coxeter matroid polytope, see Appendix B.)

3.5. **Gelfand-Serganova Theorem.** We finally present the theorem which is the keystone to the whole theory of Coxeter matroids.

Theorem 3.5.1. *Let W be a finite Coxeter group, P a parabolic subgroup in W, $\mathcal{M}$ a subset in W^P, and $\Delta_{\mathcal{M}}$ the polytope associated with $\mathcal{M}$.*

Then the following conditions are equivalent:

(1) $\mathcal{M}$ is a Coxeter matroid.

(2) $\Delta_{\mathcal{M}}$ is convex and every edge of $\Delta_{\mathcal{M}}$ is parallel to a root in Φ.

A useful reformulation of the Gelfand-Serganova Theorem is as follows:

[7]A *polyhedron* is the intersection of a finite number of closed half-spaces. Since half-spaces are convex, a polyhedron is also convex. A bounded polyhedron is called a *polytope*.

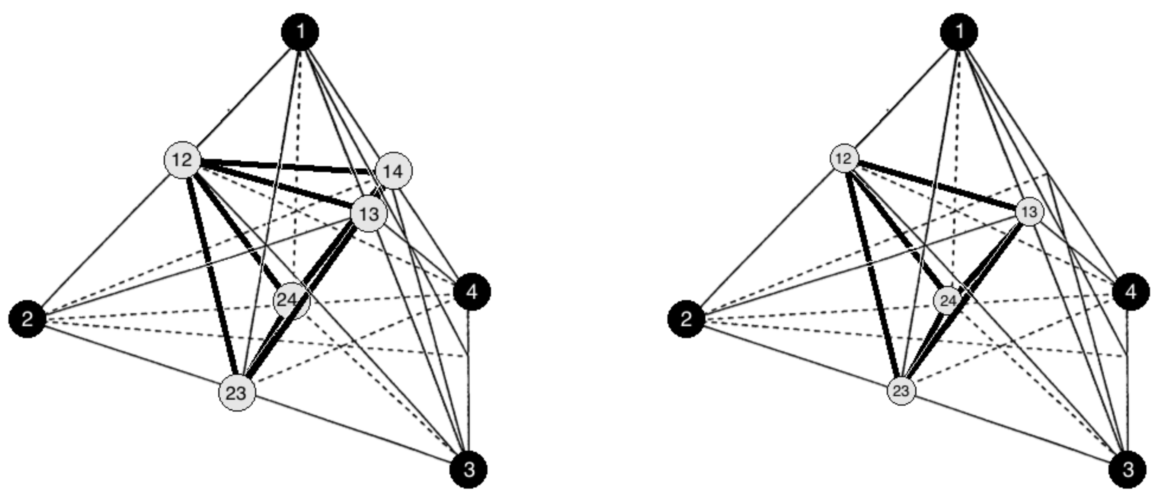
Theorem 3.5.2. *A subset $\mathcal{M} \subseteq W^P$ is a Coxeter matroid if and only if, for any pair of adjacent vertices δ_A and δ_B of $\Delta_{\mathcal{M}}$, there exists a reflection $s \in W$ such that $s\delta_A = \delta_B$ (and also $s\delta_B = \delta_A$, $sB = A$, and $sA = B$).*

4. Examples

4.1. **A Coxeter matroid polytope for $W = S_4$.** Consider the previous example $\mathcal{M} = \{12, 13, 14, 23, 24\} \subseteq W^P$ from Section 2.4.1, where $W = S_4$ and $P = W_{\{1,3\}}$. The corresponding polytope $\Delta_{\mathcal{M}}$ is shown in Fig. 5a. It can be easily checked that every edge of $\Delta_{\mathcal{M}}$ is parallel to a root in Φ (recall that the roots are parallel to the edges of the simplex), and hence by the Gelfand-Serganova theorem, $\mathcal{M}$ is a Coxeter matroid.

4.2. **A non-example.** For $W = S_4$ and the same maximal standard parabolic subgroup $P = W_{\{1,3\}}$ from Section 2.4.1, consider the subset $\mathcal{M} = \{12, 13, 23, 24\} \subseteq W^P$. The corresponding polytope $\Delta_{\mathcal{M}}$ is shown in Fig. 5b. Clearly, $\{13, 24\}$ is an edge of $\Delta_{\mathcal{M}}$ which is not parallel to any edge of the simplex, and therefore $\mathcal{M}$ is not a Coxeter matroid. Or alternatively, since the edge $\{13, 24\}$ is not parallel to any edge of the polytope, we can easily see that there is no reflection in W that sends $\{1, 3\}$ to $\{2, 4\}$, and hence by Theorem 3.5.2, $\mathcal{M}$ is not a Coxeter matroid.

Additionally, it can be checked using the matroid basis axioms that the collection $\mathcal{M}$ is not the collection of bases of an ordinary matroid of rank 2 on the set $E = [4]$.



(A) The Coxeter matroid polytope of Example 2.4.1 (B) The non-matroid polytope of Example 4.2

Figure 5. The polytopes from Sections 4.1 and 4.2, for $W = S_4$ and $P = W_{\{1,3\}} = \langle s_1, s_3 \rangle$

4.3. **A symplectic matroid polytope.** Take $W = BC_3$, and let $P = W_{\{1,3\}} = \langle s_1, s_3 \rangle$ be a maximal standard parabolic subgroup of BC_3. We can think of the permutations in BC_3 as permuting the word $1233^*2^*1^*$, and hence

$$P = \langle (1,2)(1^*, 2^*), (3, 3^*) \rangle = \{1233^*2^*1^*, 2133^*1^*2^*, 123^*32^*1^*, 213^*31^*2^*\}.$$

Similarly to Example 4.3 above, every left coset of P corresponds to a 2-element subset of E, namely, the two elements which appear (in some order) as the first two symbols in each word of the coset. For example,

$$\{1, 3\} \quad \leftrightarrow \quad (23)(2^*3^*)P = \{1322^*3^*1^*, 3122^*1^*3^*, 132^*23^*1^*, 312^*21^*3^*\}.$$

Let $\mathcal{M} = \{12, 13, 1^*2, 1^*3^*, 23, 23^*\} \subseteq W^P$ where each pair listed corresponds to a left coset of P, namely

$$\begin{array}{rcll}
\{1,2\} & \leftrightarrow & P & = \{1233^*2^*1^*, 2133^*1^*2^*, 123^*32^*1^*, 213^*31^*2^*\} \\
\{1,3\} & \leftrightarrow & (23)(2^*3^*)P & = \{1322^*3^*1^*, 3122^*1^*3^*, 132^*23^*1^*, 312^*21^*3^*\} \\
\{1^*,2\} & \leftrightarrow & (33^*)(11^*)P & = \{1^*23^*32^*1, 21^*3^*312^*, 1^*233^*2^*1, 21^*33^*12^*\} \\
\{1^*,3^*\} & \leftrightarrow & (13^*2^*)(1^*32)P & = \{3^*1^*22^*13, 1^*3^*22^*31, 3^*1^*2^*213, 1^*3^*2^*231\} \\
\{2,3\} & \leftrightarrow & (13)(1^*3^*)P & = \{3211^*2^*3^*, 2311^*3^*2^*, 321^*12^*3^*, 231^*13^*2^*\} \\
\{2,3^*\} & \leftrightarrow & (13^*)(1^*3)P & = \{3^*21^*12^*3, 23^*1^*132^*, 3^*211^*2^*3, 23^*11^*32^*\}.
\end{array}$$

The corresponding polytope $\Delta_{\mathcal{M}}$ is given in Fig. 6. It can be checked that every edge of $\Delta_{\mathcal{M}}$ is parallel to a root in Φ, and hence by the Gelfand-Serganova theorem, $\mathcal{M}$ is a Coxeter matroid.

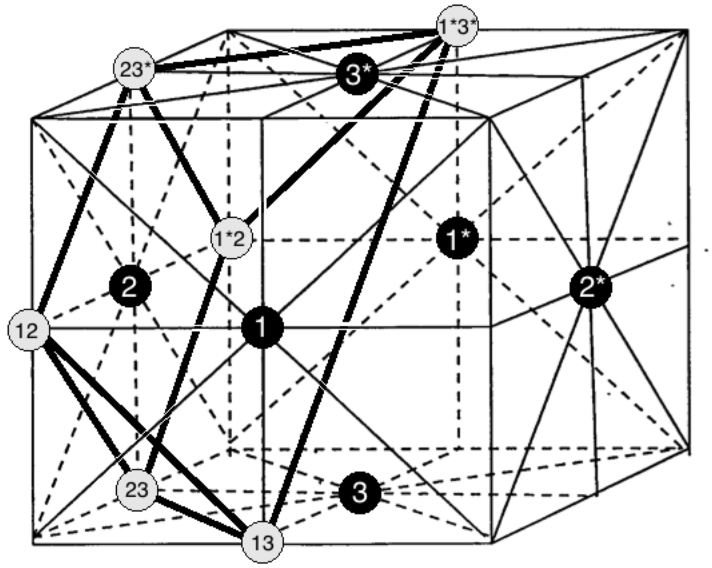


FIGURE 6. The symplectic matroid polytope of Example 4.3

In similar fashion, we may check that the left cosets of maximal parabolic subgroups $P = W_I$ correspond to $\xi(I)$-element subsets of E, and hence to barycenters of $n - \xi(I)$-dimensional faces of the n-cube.

Appendix A. Ordinary matroids

One of the many equivalent definitions of an ordinary matroid is the following:

Definition. A *matroid* is a pair $(E, \mathcal{B})$ where E is a finite set and $\mathcal{B}$ is a collection of subsets of E (called the "bases" of the matroid), with the following properties:

(1) $\mathcal{B} \neq \emptyset$,

(2) (Basis exchange property) For all $A, B \in \mathcal{B}$ and $a \in A \backslash B$, there exists an element $b \in B \backslash A$ such that $A \backslash \{a\} \cup \{b\} \in \mathcal{B}$.

Example. An example of a matroid of rank 2 on the set $E = [4] = \{1,2,3,4\}$ would be $(E, \mathcal{B})$ with $\mathcal{B} = \{12, 13, 14, 23, 24\}$, where ab is an abbreviation of the two-element subset $\{a, b\}$ of $E = [n]$.

Appendix B. Coxeter matroid polytope

Let Δ be a convex polytope in the real affine Euclidean space $\mathbb{AR}^n$. For any two vertices α and β of Δ which are adjacent (i.e., connected by an edge) we can consider the reflection $s_{\alpha\beta}$ in the mirror of symmetry of the edge $[\alpha\beta]$. All these reflections generate a group $W(\Delta)$ of affine isometries of the space $\mathbb{AR}^n$. We say that Δ is a *Coxeter matroid polytope* if the group $W(\Delta)$ is finite.

If Δ is a matroid polytope, then W is a finite reflection group and hence a Coxeter group. Being a finite group, W fixes the barycenter of each of its (finite) orbits, so we may assume without loss of generality that W fixes the origin of the vector space $\mathbb{R}^n$ and hence is a linear group. By definition of W all vertices of Δ belong to one W-orbit. Choose a vertex δ of Δ, and let E be a chamber such that δ belongs to the closure $\overline{E}$ of E. Choose the simple system Π with corresponding system of simple reflections $r_1, ..., r_m$ such that E is the fundamental chamber with respect to Π. It follows from ([1], Theorem 5.5.1) that the isotropy group P of δ is a standard parabolic subgroup of W, i.e., it is generated by some r_i's.

Therefore, the set of vertices of Δ can be identified with some subset $\mathcal{M}$ of the factor set W^P. Now the following result is an immediate corollary of Theorem 3.5.2:

Theorem B.0.1. *If Δ is a matroid polytope, then $\mathcal{M}$ is a Coxeter matroid for W and P.*

By the Gelfand-Serganova theorem, the converse is also true: if $\mathcal{M}$ is a Coxeter matroid, then its canonical matroid polytope $\Delta_{\mathcal{M}}$ is a Coxeter matroid polytope.

References

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